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# Second order three boundary value problem in Banach spaces via Henstock and Henstock–Kurzweil–Pettis integral

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**Abstract**

Existence theorems and some properties of solutions set of three boundary value second order differential equations and inclusions in Banach spaces are obtained under Henstock, respectively Henstock–Kurzweil–Pettis integrability assumptions. Our results extend those obtained by Azzam, Castaing and Thibault in the Bochner integrability setting and in the Pettis integrability one. The continuity of the (unique) solution with respect to a parameter in the single-valued case is also studied.


*Keywords:* Differential inclusion; Henstock integral; Henstock–Kurzweil–Pettis (set-valued) integral; Hartman function

## 1. Introduction

Second order differential equations and inclusions with three boundary conditions were studied in finite-dimensional setting (e.g. in [10]) and then in the general context of Banach spaces in [1], using Hartman-type functions. Such a function was introduced for the first time in [9] for the study of second order differential equation with two boundary conditions.

The problem we investigate here is the second order differential inclusion

$$(*)\qquad \begin{cases} \ddot{u}(t) \in F\big(t, u(t), \dot{u}(t)\big), & \text{a.e. } t \in [0,1],\\ u(0) = 0,\ u(\theta) = u(1), \end{cases}$$

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where $F : [0, 1] \times X \times X \to \mathcal{P}_{wkc}(X)$ ($X$ being a separable Banach space) and $\theta \in ]0, 1[$. Existence results were given and properties of the set of solutions were investigated in [1] under integrable boundedness assumptions in the Bochner integrability setting (see Theorem 2.2), respectively under scalar Pettis uniform integrability assumptions in the Pettis integrability case (Theorem 3.3).

In the present paper, we extend these results to more general cases of Henstock integrably bounded, respectively Henstock–Kurzweil–Pettis integrable multifunctions. Sobolev-type spaces appropriate to our setting are involved.

Finally, in the single-valued case, we provide sufficient conditions for the function that is governing the equation to ensure that the unique solution is continuous with respect to a parameter.

## 2. Notations and preliminary facts

We begin by introducing the basic facts on Henstock–Kurzweil integrability, a concept that extends the classical Lebesgue integrability on the real line.

Let $[0, 1]$ be the real unit interval provided with the $\sigma$-algebra $\Sigma$ of Lebesgue measurable sets and with the Lebesgue measure $\mu$. A gauge $\delta$ on $[0, 1]$ is a positive function. A partition of $[0, 1]$ is a finite family $(I_i, t_i)_{i=1}^n$ of nonoverlapping intervals covering $[0, 1]$ with tags $t_i \in I_i$; a partition is said to be $\delta$-fine if for each $i \in \{1, \dots, n\}$, $I_i \subset ]t_i - \delta(t_i), t_i + \delta(t_i)[$. A function $f : [0, 1] \to \mathbb{R}$ is said to be Henstock–Kurzweil (shortly HK-) integrable if there exists a real, denoted by $(\mathrm{HK}) \int_0^1 f(t)\,dt$, satisfying that, for every $\varepsilon > 0$, there is a gauge $\delta_\varepsilon$ such that $|\sum_{i=1}^n f(t_i)\mu(I_i) - (\mathrm{HK}) \int_0^1 f(t)\,dt| < \varepsilon$, for every $\delta_\varepsilon$-fine partition $\mathcal{P}=(I_i, t_i)_{i=1}^n$ of $[0, 1]$. For properties of HK-integral, we refer the reader to [8].

Let us recall the following integration by parts result [8, Theorem 12.8]:

**Lemma 1.** *Let $f : [a, b] \to \mathbb{R}$ be HK-integrable and $g : [a, b] \to \mathbb{R}$ be absolutely continuous. Then $fg$ is HK-integrable and, for every $t \in [a, b]$,*

$$(\mathrm{HK}) \int_a^t f(s)g(s)\,ds = g(t)(\mathrm{HK}) \int_a^t f(s)\,ds - \int_a^t \left( g'(s)(\mathrm{HK}) \int_a^s f(\tau)\,d\tau \right) ds.$$

**Remark 2.** The second term in the right-hand side is a Lebesgue integral. Moreover, a similar property is valid for a bounded variation function $g$ (not necessarily absolutely continuous), but in that case a Riemann–Stieltjes integral has to be considered in the right-hand side (see [8, Theorem 12.21]).

Denote by $\mathcal{HK}([0, 1])$ the space of all HK-integrable functions provided with the Alexiewicz norm: $\|f\|_A = \sup_{[a,b]\subset[0,1]} |(\mathrm{HK}) \int_a^b f(s)\,ds|$. It was proved that:

**Lemma 3.** [13] *$T$ is a linear continuous functional on $\mathcal{HK}([0, 1])$ if and only if there exists a real function $g$ of bounded variation such that, for every $f \in \mathcal{HK}([0, 1])$, $T(f) = (\mathrm{HK}) \int_0^1 f(s)g(s)\,ds$.*

**Remark 4.** The class of Henstock–Kurzweil integrable functions (which coincides with the class of Denjoy and Perron integrable functions, cf. [8]) is contained in the class of Khintchine integrable functions (see [8, Chapter 15]). In [6] and [7], the Khintchine integrability is called Denjoy integrability. This will not lead us to any confusion, because we will use only the HK-integral and, for any reference to [6] or [7], the Khintchine appellation.

Through the paper, $X$ is a real separable Banach space, $X^*$ (respectively $X^{**}$) is its topological dual (respectively bidual), $X_w$ denotes the space provided with its weak topology and $\mathcal{P}_{wkc}(X)$ (respectively $\mathcal{P}_{kc}(X)$) stands for the family of its weakly compact (respectively strongly compact) convex subsets. On $\mathcal{P}_{wkc}(X)$ the Hausdorff distance $D$ is considered and, for every $A \in \mathcal{P}_{wkc}(X)$, we put $|A| = D(A, \{0\})$.

The following notion extends the real HK-integral to the vector case.

**Definition 5.** A function $f : [0,1] \to X$ is Henstock integrable if there exists $\tilde{f} : [0,1] \to X$ such that, for every $\varepsilon > 0$, there is a gauge $\delta_\varepsilon > 0$ satisfying that $\sum_{i=1}^{n} \| f(t_i)\mu(I_i) - [\tilde{f}(x_i) - \tilde{f}(x_{i-1})]\| < \varepsilon$, for each $\delta_\varepsilon$-fine partition $\mathcal{P} = (I_i, t_i)_{i=1}^{n}$, where $x_{i-1}$ and $x_i$ are the extremities of the interval $I_i$.

We will use the notation (H) $\int_0^t f(s)\,ds = \tilde{f}(t)$.

A Henstock integrable function is Henstock integrable on every subinterval. Any Bochner integrable function is Henstock-integrable. As the following theorem states, the Henstock integrable Banach-valued functions possess (like the Bochner integrable ones) an important property of differentiability.

**Theorem 6.** ([2] or [8] for $X = \mathbb{R}$.) *Let $f : [0,1] \to X$ be Henstock-integrable. Then $\tilde{f}$ is continuous, a.e. derivable and $(\tilde{f})'(t) = f(t)$ a.e.*

Another well-known extension of Lebesgue integral to the Banach-valued case is the Pettis integral (see [11]). It can be generalized by considering, on the real line, the HK-integral instead of the Lebesgue one, as follows:

**Definition 7.** A function $f : [0,1] \to X$ is said to be Henstock–Kurzweil–Pettis (shortly HKP-) integrable if:

(1) $f$ is scalarly HK-integrable, i.e. for all $x^* \in X^*$, $\langle x^*, f(\cdot)\rangle$ is HK-integrable;
(2) for each $[a,b] \subset [0,1]$, there exists $x_{[a,b]} \in X$ such that, for all $x^* \in X^*$, $\langle x^*, x_{[a,b]}\rangle =$ (HK) $\int_a^b \langle x^*, f(t)\rangle\,dt$. We denote $x_{[a,b]} =$ (HKP) $\int_a^b f(t)\,dt$.

If the condition (2) requires only $x_{[a,b]} \in X^{**}$, then $f$ is called Henstock–Kurzweil–Dunford (shortly HKD-) integrable.

**Remark 8.**

(i) Example 42 in [7] contains a Pettis integrable function that is not Bochner integrable on any subinterval of $[0,1]$, therefore it is not Henstock integrable. Reciprocally, a Henstock integrable function is not necessarily Pettis integrable.
(ii) Obviously, any Henstock integrable function is HKP-integrable. The converse is not true: the function considered in Example 1 in [12] (due to [6]) is HKP-integrable, but not Henstock integrable.

(iii) Any Pettis integrable function is HKP-integrable; the converse is not true: a counterexample is given by the above mentioned function from [12].

We can consider (via Lemma 3) the space of all HKP-integrable $X$-valued functions provided with the topology induced by the tensor product of the space of real functions of bounded variation and $X^*$ (we call it the weak-Henstock–Kurzweil–Pettis topology and denote it by w-HKP). That is: $f_\alpha \to f$ if, for every $g:[0,1] \to \mathbb{R}$ of bounded variation and every $x^* \in X^*$, (HK) $\int_0^1 g(s)\langle x^*, f_\alpha(s)\rangle\,ds \to$ (HK) $\int_0^1 g(s)\langle x^*, f(s)\rangle\,ds$. Our consideration arises naturally from the Pettis integrability setting, where the weak-Pettis topology is the one induced by the tensor product $L^\infty([0,1]) \otimes X^*$.

Through this paper, we will use the following notion of weak derivability:

**Definition 9.** A function $G:[0,1] \to X$ is said to be weakly derivable with the weak derivative $g$ if, for every $x^* \in X^*$, there exists $N(x^*) \subset [0,1]$ of null measure such that $\langle x^*, G\rangle$ is derivable on $[0,1] \setminus N(x^*)$ and its derivative is $\langle x^*, G(t)\rangle' = \langle x^*, g(t)\rangle$, for every $t \in [0,1] \setminus N(x^*)$.

Since the Banach space is separable, the weak derivative, if it exists, is unique up to a null measure set. Indeed, let $g_1$, $g_2$ be two weak derivatives of $G$ and $(x_n^*)_n$ a sequence of $B^*$ which separates the points of $X$. For each $n$, we can find $N_n \subset [0,1]$ of null measure such that $\langle x_n^*, g_1(t)\rangle = \langle x_n^*, G(t)\rangle' = \langle x_n^*, g_2(t)\rangle$, for every $t \in [0,1] \setminus N_n$. Denoting by $N = \bigcup_{n\in\mathbb{N}} N_n$, we obtain that $\mu(N) = 0$ and $\langle x_n^*, g_1(t)\rangle = \langle x_n^*, g_2(t)\rangle$, for all $t \in [0,1] \setminus N$ and $n \in \mathbb{N}$. It follows that $g_1(t) = g_2(t)$, for every $t \in [0,1] \setminus N$.

**Remark 10.** By Theorem 6 it follows that, if $f$ is HKP-integrable, then its primitive (HKP) $\int_0^\cdot f(s)\,ds$ is weakly continuous and weakly derivable and its weak derivative is (a.e. equal to) $f$.

We denote the support functional of $A \in \mathcal{P}_{wkc}(X)$ by $\sigma(\cdot, A)$. A function $f:[0,1] \to X$ is a selection of $\Gamma$ if $f(t) \in \Gamma(t)$ a.e. For all concepts of measurability, we refer the reader to [4]. A multifunction $\Gamma$ is said to be:

(i) integrably bounded if the real function $|\Gamma(\cdot)|$ is Lebesgue integrable;
(ii) scalarly HK-integrable if, for every $x^* \in X^*$, $\sigma(x^*, \Gamma(\cdot))$ is HK-integrable;
(iii) A $\mathcal{P}_{wkc}(X)$-valued function $\Gamma$ is "HKP-integrable in $\mathcal{P}_{wkc}(X)$" (shortly HKP-integrable) if it is scalarly HK-integrable and for every $[a,b] \subset [0,1]$, there exists (HKP) $\int_a^b \Gamma(t)\,dt \in \mathcal{P}_{wkc}(X)$ such that, for every $x^* \in X^*$, $\sigma(x^*, \text{(HKP)} \int_a^b \Gamma(t)\,dt) =$ (HK) $\int_a^b \sigma(x^*, \Gamma(t))\,dt$.

Obviously, in the particular case of a single-valued function, these concepts coincide with those previously given in the vector case.

Let us recall following characterizations of HKP-integrable multifunctions (for definition and properties of Pettis set-valued integral we refer to [5]):

**Theorem 11.** [12, Theorem 1] *Let $\Gamma:[0,1] \to \mathcal{P}_{wkc}(X)$ be scalarly HK-integrable. Then the following conditions are equivalent*:

(i) *$\Gamma$ is HKP-integrable*;

(ii) *$\Gamma$ has at least one HKP-integrable selection and for any HKP-integrable selection $f$ there exists a Pettis integrable multifunction $\Gamma_1 : [0,1] \to \mathcal{P}_{wkc}(X)$ such that, for every $t \in [0,1]$, $\Gamma(t) = f(t) + \Gamma_1(t)$;*

(iii) *each measurable selection of $\Gamma$ is HKP-integrable.*

## 3. Three boundary value second order differential inclusion via Henstock–Kurzweil–Pettis integral

Denote by $W^{2,1}_{\mathrm{HKP},X}([0,1])$ the set of all functions $u : [0,1] \to X$ that are weakly continuous, weakly derivable with the weak derivative $\dot{u}$ weakly continuous and weakly derivable and the second weak derivative $\ddot{u}$ HKP-integrable.

In the study of three boundary value second order differential inclusions, we will use a Hartman-type function. Such a function was first considered in the study of two boundary problems for ordinary differential equations in [9]. Consider $G : [0,1] \times [0,1] \to \mathbb{R}$ the Hartman function introduced in [1]:

$$\text{if } 0 \leqslant t < \theta, \quad G(t,s) = \begin{cases} -s, & \text{if } 0 \leqslant s \leqslant t, \\ -t, & \text{if } t < s \leqslant \theta, \\ \frac{t(s-1)}{1-\theta}, & \text{if } \theta < s \leqslant 1, \end{cases}$$

and

$$\text{if } \theta \leqslant t \leqslant 1, \quad G(t,s) = \begin{cases} -s, & \text{if } 0 \leqslant s < \theta, \\ \frac{\theta(s-t)+s(t-1)}{1-\theta}, & \text{if } \theta \leqslant s \leqslant t, \\ \frac{t(s-1)}{1-\theta}, & \text{if } t < s \leqslant 1. \end{cases}$$

In [1] it is proved that $G(\cdot,s)$ is derivable, for every $s \in [0,1]$:

$$\text{if } 0 \leqslant t < \theta, \quad \frac{\partial G}{\partial t}(t,s) = \begin{cases} 0, & \text{if } 0 \leqslant s < t, \\ -1, & \text{if } t < s \leqslant \theta, \\ \frac{s-1}{1-\theta}, & \text{if } \theta < s \leqslant 1, \end{cases}$$

and

$$\text{if } \theta \leqslant t \leqslant 1, \quad \frac{\partial G}{\partial t}(t,s) = \begin{cases} 0, & \text{if } 0 \leqslant s \leqslant \theta, \\ \frac{s-\theta}{1-\theta}, & \text{if } \theta < s < t, \\ \frac{s-1}{1-\theta}, & \text{if } t < s \leqslant 1. \end{cases}$$

It can easily be seen that

**Lemma 12.** *For every $t \in [0,1]$, $G(t,\cdot)$ and $\frac{\partial G}{\partial t}(t,\cdot)$ are derivable on $[0,t[$, $]t,\theta[$ and $]\theta,1]$ (if $t < \theta$), respectively on $[0,\theta[$, $]\theta,t[$ and $]t,1]$ (if $\theta \leqslant t$) and their derivatives are absolutely continuous.*

One can use the Hartman-type function $G$ in order to obtain elements of $W^{2,1}_{\mathrm{HKP},X}([0,1])$:

**Proposition 13.** *Let $f : [0,1] \to X$ be a HKP-integrable function. Then*:

(1) *for every $t \in [0,1]$, $G(t,\cdot)f(\cdot)$ and $\frac{\partial G}{\partial t}(t,\cdot)f(\cdot)$ are HKP-integrable and the function*

$$u_f : [0,1] \to X, \quad u_f(t) = (\mathrm{HKP})\int_0^1 G(t,s)f(s)\,ds, \quad \forall t \in [0,1],$$

*satisfies the following conditions*: $u_f(0)=0$, $u_f(\theta)=u_f(1)$; *and*

(2) $u_f$ *is weakly continuous*;

(3) $u_f$ *is weakly derivable*; *its weak derivative is* $\dot{u}_f(t)=(\mathrm{HKP})\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds$;

(4) $\dot{u}_f$ *is weakly derivable and* $\ddot{u}_f=f$.

**Proof.** We will consider only the case $t\in[0,\theta[$ (the proof in the other case being similar). For every $t\in[0,\theta[$, the function $G(t,\cdot)$ is absolutely continuous. The same is true for $\frac{\partial G}{\partial t}(t,\cdot)$ on $[0,t[$ and $]t,1]$. Applying Lemma 1 we obtain that, for each $x^*\in X^*$, the real functions $G(t,\cdot)\langle x^*,f(\cdot)\rangle$ and $\frac{\partial G}{\partial t}(t,\cdot)\langle x^*,f(\cdot)\rangle$ are HK-integrable on [0, 1]. Since the integrability in HK sense is stronger than the integrability in Khintchine sense (see Remark 4), cf. [6, Theorem 3], $G(t,\cdot)f(\cdot)$ and $\frac{\partial G}{\partial t}(t,\cdot)f(\cdot)$ are HKP-integrable.

By Lemma 1 it follows that, for every $x^*\in X^*$ and every $s\in[0,1]$,

$$(\mathrm{HK})\int_0^s \langle x^*, G(t,\tau)f(\tau)\rangle\,d\tau$$
$$=G(t,s)(\mathrm{HK})\int_0^s\langle x^*,f(\tau)\rangle\,d\tau-\int_0^s\frac{\partial G}{\partial \tau}(t,\tau)\left\langle x^*,(\mathrm{HKP})\int_0^\tau f(\rho)\,d\rho\right\rangle d\tau,$$

whence

$$(\mathrm{HKD})\int_0^s G(t,\tau)f(\tau)\,d\tau$$
$$=G(t,s)(\mathrm{HKD})\int_0^s f(\tau)\,d\tau-\int_0^s\left(\frac{\partial G}{\partial \tau}(t,\tau)(\mathrm{HKD})\int_0^\tau f(\rho)\,d\rho\right)d\tau.$$

By hypothesis, the first term of the sum is an element of $X$. The second term is an element of $X$ since, by [8, Theorem 9.12] and Lemma 12, the function $\tau\mapsto\frac{\partial G}{\partial \tau}(t,\tau)(\mathrm{HKP})\int_0^\tau f(\rho)\,d\rho$ is bounded and measurable, so it is Bochner integrable. Then, for every $t\in[0,\theta[$, $G(t,\cdot)f(\cdot)$ is HKP-integrable and, in the same way, we can prove the HKP-integrability of $\frac{\partial G}{\partial t}(t,\cdot)f(\cdot)$.

By definition, the $X$-valued function defined by $u_f(t)=(\mathrm{HKP})\int_0^1 G(t,s)f(s)\,ds$ satisfies the conditions

$$u_f(\theta)=\int_0^\theta -sf(s)\,ds+\frac{\theta}{1-\theta}\int_\theta^1(s-1)f(s)\,ds=u_f(1)\quad\text{and}$$

$$u_f(0)=(\mathrm{HKP})\int_0^1 G(0,s)f(s)\,ds=0.$$

In order to prove the assertions (2)–(4), let us fix $x^*\in X^*$. We have then

$$\langle x^*,u_f(t)\rangle=(\mathrm{HK})\int_0^1\langle x^*,G(t,s)f(s)\rangle\,ds$$

$$= (\mathrm{HK}) \int_0^t \big\langle x^*, -sf(s)\big\rangle\, ds - t(\mathrm{HK}) \int_t^\theta \langle x^*, f(s)\rangle\, ds$$

$$+ t(\mathrm{HK}) \int_\theta^1 \Big\langle x^*, \frac{s-1}{1-\theta} f(s)\Big\rangle\, ds.$$

The first two terms are continuous and a.e. derivable by Theorem 6, while the last one is linear in $t$.

Its (a.e.) derivative is equal to

$$\big\langle x^*, u_f(t)\big\rangle' = \left\langle x^*, (\mathrm{HKP}) \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s)\, ds\right\rangle.$$

That means that $u_f$ is weakly derivable and its weak derivative is given by

$$\dot{u}_f(t) = (\mathrm{HKP}) \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s)\, ds$$

and it is weakly continuous. Again by Theorem 6, for every $x^* \in X^*$, $\langle x^*, \dot{u}_f(t)\rangle$ is a.e. derivable and its derivative is $\langle x^*, \dot{u}_f(t)\rangle' = \langle x^*, f(t)\rangle$. Consequently, $\ddot{u}_f(t) = f(t)$. □

An immediate consequence is the following existence and uniqueness result:

**Proposition 14.** *Let $f : [0,1] \to X$ be a HKP-integrable function. Then the second order differential equation*

$$\begin{cases} \ddot{u}(t) = f(t), & a.e.\ t \in [0,1], \\ u(0) = 0,\ u(\theta) = u(1), \end{cases}$$

*has a unique $W^{2,1}_{\mathrm{HKP},X}([0,1])$-solution, $u_f(t) = (\mathrm{HKP}) \int_0^1 G(t,s) f(s)\, ds$.*

In order to obtain the existence of solutions of differential inclusion $(*)$, we can make use of Kakutani–Ky Fan's fixed point theorem (as in [1]). Although we will use another method of proof, we think that it is worthwhile to give some auxiliary results concerning HKP set-valued integration, which can be of a larger interest:

**Lemma 15.** *Let $\Gamma : [0,1] \to \mathcal{P}_{wkc}(X)$ be HKP-integrable. The undefined HKP set-valued integral is upper semi-continuous with respect to the weak topology.*

**Proof.** For each $x^* \in X^*$, the real function $\sigma(x^*, \Gamma(\cdot))$ is HK-integrable. Theorem 6 yields that, for every $\varepsilon > 0$, we can find $\delta_{x^*,\varepsilon}$ such that, for all $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_{x^*,\varepsilon}$, $|(\mathrm{HK}) \int_{t_1}^{t_2} \sigma(x^*, \Gamma(s))\, ds| \leqslant \varepsilon$, whence $|\sigma(x^*, (\mathrm{HKP}) \int_{t_1}^{t_2} \Gamma(s)\, ds)| \leqslant \varepsilon$. □

**Remark 16.** As the HKP-integrals are weakly compact, by [4, Theorem II-25], $\bigcup_{[a,b]\subset[0,1]} (\mathrm{HKP}) \int_a^b \Gamma(s)\, ds$ is weakly compact, whence, by [12, Theorem 1], the set $\{(\mathrm{HKP}) \int_a^b f(s)\, ds,\ f$ HKP-integrable selection of $\Gamma$, $[a,b] \subset [0,1]\}$ is weakly compact.

**Corollary 17.** *Let $\Gamma : [0,1] \to \mathcal{P}_{wkc}(X)$ be HKP-integrable. Then the family of undefined HKP-integrals of all HKP-integrable selections of $\Gamma$ is equi-uniformly continuous with respect to the weak topology on $X$.*

A weak compactness result on the family of integrable selections similar to those already known in Bochner and Pettis integrability settings can be proved:

**Proposition 18.** *Let $\Gamma : [0,1] \to \mathcal{P}_{wkc}(X)$ be HKP-integrable. Then the family of all HKP-integrable selections of $\Gamma$ is sequentially w-HKP compact.*

**Proof.** Applying Theorem 11, we can find a HKP-integrable function $\gamma$ and a $\mathcal{P}_{wkc}(X)$-valued Pettis integrable multifunction $\Gamma_1$ such that $\Gamma(t) = \gamma(t) + \Gamma_1(t)$, for any $t \in [0,1]$. Let $(f_n)_n$ be a sequence of HKP-integrable selections of $\Gamma$. For every $n \in \mathbb{N}$, there exists a Pettis integrable selection of $\Gamma_1$, denoted by $g_n$, such that $f_n(t) = \gamma(t) + g_n(t)$, for every $t \in [0,1]$. By [3, Proposition 3.4], we can find a subsequence $(g_{k_n})_n$ that converges with respect to the weak-Pettis topology to a Pettis integrable selection $g$ of $\Gamma_1$. It follows that $(f_{k_n})_n$ w-HKP-converges to $g + \gamma$ that is a selection of $\Gamma$. $\square$

The following generalizes the first part of Lemma 1 to set-valued HKP-integral:

**Lemma 19.** *Let $\Gamma : [0,1] \to \mathcal{P}_{wkc}(X)$ be a HKP-integrable multifunction and $g : [0,1] \to \mathbb{R}$ be absolutely continuous. Then $g\Gamma$ is HKP-integrable.*

**Proof.** Let $\gamma$ be a HKP-integrable function and $\Gamma_1$ a $\mathcal{P}_{wkc}(X)$-valued Pettis integrable multifunction such that $\Gamma(s) = \gamma(s) + \Gamma_1(s)$, for any $s \in [0,1]$. Then $g(s)\Gamma(s) = g(s)\gamma(s) + g(s)\Gamma_1(s)$. By Lemma 1, the function $s \mapsto g(s)\gamma(s)$ is HKP-integrable and, since $g$ is absolutely continuous, by using the characterization of $\mathcal{P}_{wkc}(X)$-valued Pettis integrable multifunctions [5, Theorem 5.4], $s \mapsto g(s)\Gamma_1(s)$ is Pettis integrable. Applying Theorem 11, we obtain that $s \mapsto g(s)\Gamma(s)$ is HKP-integrable. $\square$

**Proposition 20.** *Let $\Gamma : [0,1] \to \mathcal{P}_{wkc}(X)$ be a HKP-integrable set-valued function. Then the $W^{2,1}_{\mathrm{HKP},X}([0,1])$-solution set of the second order differential inclusion*

$$\begin{cases} \ddot{u}(t) \in \Gamma(t), & \text{a.e. } t \in [0,1], \\ u(0) = 0, \ u(\theta) = u(1) \end{cases}$$

*is nonempty, convex and compact in $C([0,1], X_w)$ provided with the topology of the uniform convergence. Moreover, if a sequence $(u_n)_n$ of solutions converges uniformly to $u$, then the sequence $(\dot{u}_n)_n$ converges weakly pointwise to $\dot{u}$ and $(\ddot{u}_n)_n$ converges to $\ddot{u}$ with respect to the w-HKP topology.*

**Proof.** By Proposition 13, any $W^{2,1}_{\mathrm{HKP},X}([0,1])$-solution $u$ of our inclusion is characterized by the existence of a HKP-integrable selection $f$ of $\Gamma$ such that

$$u(t) = u_f(t) = (\mathrm{HKP}) \int_0^1 G(t,s) f(s)\, ds, \quad \forall t \in [0,1].$$

Let us note that, by Theorem 11, the set of solutions is nonempty and it is, obviously, convex, since $\Gamma$ is convex-valued.

In order to prove the compactness of the solution set, we will make use of Ascoli's Theorem. Let us begin by proving the equicontinuity.

By Lemma 19, the $\mathcal{P}_{wkc}(X)$-valued function $s \mapsto s\Gamma(s)$ is HKP-integrable.

Fix $x^* \in X^*$ and $\varepsilon > 0$. Remark 16 yields that

$$M = \sup\left\{\max\left(\left|\int_a^b \langle x^*, f(s)\rangle\,ds\right|, \left|\int_a^b \langle x^*, sf(s)\rangle\,ds\right|\right)\right\} < +\infty,$$

the supremum being taken over all HKP-integrable selections of $\Gamma$ and all intervals $[a,b] \subset [0,1]$. Corollary 17 allows us to choose $\delta_{x^*,\varepsilon} > 0$ such that, for every HKP-integrable selection $f$ of $\Gamma$ and every $t_1, t_2$ with $|t_1 - t_2| < \delta_{x^*,\varepsilon}$,

$$\max\left(\left|(\mathrm{HK})\int_{t_1}^{t_2} \langle x^*, f(s)\rangle\,ds\right|, \left|(\mathrm{HK})\int_{t_1}^{t_2} \langle x^*, sf(s)\rangle\,ds\right|\right) < \eta_\varepsilon.$$

Choosing conveniently $\delta_{x^*,\varepsilon}$ and $\eta_\varepsilon$ and considering the three possible cases ($t_1 < t_2 < \theta$, $t_1 < \theta \leqslant t_2$ and $\theta \leqslant t_1 < t_2$) we obtain $|\langle x^*, u_f(t_1) - u_f(t_2)\rangle| \leqslant \varepsilon$, thus the equicontinuity is proved.

Fix now $t \in [0,1]$. Again by Lemma 19, $G(t,\cdot)\Gamma(\cdot)$ is HKP-integrable. For every solution $u = u_f$ of our inclusion (where $f$ is a HKP-integrable selection of $\Gamma$),

$$u(t) = u_f(t) = (\mathrm{HKP})\int_0^1 G(t,s)f(s)\,ds \in (\mathrm{HKP})\int_0^1 G(t,s)\Gamma(s)\,ds$$

that is, by definition, weakly compact and convex.

It remains us to prove only the closeness of the solution set in $C([0,1], X_w)$. Its topology is metrizable, so it suffices to consider a sequence $(u_n)_n$ of solutions, uniformly convergent to $u \in C([0,1], X_w)$, and prove that $u$ is a solution too.

We can find a sequence $(f_n)_n$ of HKP-integrable selections of $\Gamma$ such that $u_n(t) = u_{f_n}(t) = (\mathrm{HKP})\int_0^1 G(t,s)f_n(s)\,ds$, for every $t \in [0,1]$. By Proposition 18, we are able to extract a subsequence $(f_{k_n})_n$ which w-HKP converges to a HKP-integrable selection $f$ of $\Gamma$ such that the sequence $s \mapsto sf_{k_n}(s)$ be w-HKP convergent to the function $s \mapsto sf(s)$. By considering again the two possible cases ($t \in [0,\theta[$ and $t \in [\theta,1]$), we obtain that $(u_{k_n})_n$ pointwise weakly converges to $u_f(t) = (\mathrm{HKP})\int_0^1 G(t,s)f(s)\,ds$, therefore $u_f = u$; the solution set is closed in $C([0,1], X_w)$ and thus the compactness is proved.

Similarly, we can prove that $\dot u_{f_n}(t) = (\mathrm{HKP})\int_0^1 \frac{\partial G}{\partial t}(t,s)f_n(s)\,ds$ weakly converges to $\dot u_f(t) = (\mathrm{HKP})\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds$. Since a.e. $\ddot u_{f_n} = f_n$ and $\ddot u_f = f$, the w-HKP convergence of $\ddot u_{f_n}$ to $\ddot u_f$ follows from the sequential w-HKP compactness of the set of all HKP-integrable selections of $\Gamma$. $\square$

We obtain the following existence result under Henstock–Kurzweil–Pettis integrability assumptions on the right-hand side of differential inclusion

$$(*)\qquad \begin{cases} \ddot u(t) \in F\big(t, u(t), \dot u(t)\big), & \text{a.e. } t \in [0,1],\\ u(0) = 0,\ u(\theta) = u(1). \end{cases}$$

Our result extends the existence theorem proved in [1] in the case of a Pettis integrable right-hand side.

**Theorem 21.** *Let $\Gamma : [0,1] \to \mathcal{P}_{wkc}(X)$ be a HKP-integrable set-valued function and let $F : [0,1] \times X \times X \to \mathcal{P}_{wkc}(X)$ satisfy the following conditions*:

(1) $F(t,x,y) \subset \Gamma(t)$, $\forall t \in [0,1]$, $\forall x, y \in X$;
(2) $F(\cdot, x, y)$ *is measurable, for every* $x, y \in X$;
(3) $F(t, \cdot, \cdot)$ *is upper semicontinuous on* $X_w \times X_w$, *for each* $t \in [0,1]$.

*Then the* $W^{2,1}_{\mathrm{HKP},X}([0,1])$*-solutions set of the inclusion* $(*)$ *is nonempty and compact in* $C([0,1], X_w)$.

**Proof.** Applying Theorem 11, we are able to find a HKP-integrable selection $\gamma$ of $\Gamma$ and a weakly compact convex-valued Pettis integrable multifunction $\Gamma_1$ such that $\Gamma(t) = \gamma(t) + \Gamma_1(t)$, for all $t \in [0,1]$.

Then the set-valued function

$$\begin{aligned}
&\widetilde{F} : [0,1] \times X \times X \to \mathcal{P}_{wkc}(X),\\
&\widetilde{F}(t,x,y)\\
&\quad = -\gamma(t) + F\left(t, x + (\mathrm{HKP})\int_0^1 G(t,s)\gamma(s)\,ds,\ y + (\mathrm{HKP})\int_0^1 \frac{\partial G}{\partial t}(t,s)\gamma(s)\,ds\right)
\end{aligned}$$

satisfies the following conditions:

(1) $\widetilde{F}(t,x,y) \subset \Gamma_1(t)$, $\forall t \in [0,1]$, $\forall x, y \in X$;
(2) $\widetilde{F}(\cdot, x, y)$ is measurable, for every $x, y \in X$;
(3) $\widetilde{F}(t, \cdot, \cdot)$ is upper semi-continuous on $X_w \times X_w$, for each $t \in [0,1]$.

Therefore, we are able to apply the similar of [1, Theorem 3.3] for the weak topology on the Banach space $X$ (which can be proved in the same manner, as it was noticed by the authors themselves). We obtain that the set of solutions of differential inclusion

$$\begin{cases} \ddot{v}(t) \in \widetilde{F}\big(t, v(t), \dot{v}(t)\big), & \text{a.e. } t \in [0,1],\\ v(0) = 0,\ v(\theta) = v(1) \end{cases}$$

that are continuous, two times a.e. weakly derivable, the second derivative being Pettis integrable, is nonempty and compact in $C([0,1], X_w)$ provided with the topology of uniform convergence. Therefore, by Proposition 14, we deduce that, for every solution $v$ of the previous inclusion, the function $u(t) = v(t) + (\mathrm{HKP}) \int_0^1 G(t,s)\gamma(s)\,ds$ is a solution of our differential inclusion $(*)$ and that the set of solutions is compact in $C([0,1], X_w)$. $\square$

## 4. Three boundary value second order differential inclusion via Henstock integral

Let us denote by $W^{2,1}_{\mathrm{H},X}([0,1])$ the family of all $X$-valued functions $u$ that are continuous on $[0,1]$, a.e. derivable with the derivative $\dot{u}$ continuous and a.e. derivable and the second derivative $\ddot{u}$ Henstock integrable.

We will make use of the following integration by parts result, the extension of Lemma 1 to the vector case, which can be proved in the same way as [8, Theorem 12.8]:

**Lemma 22.** *Let $f:[a,b]\to X$ be Henstock integrable and $g:[a,b]\to\mathbb{R}$ be absolutely continuous. Then $fg$ is Henstock integrable and*

$$(\mathrm{H})\int_a^t f(s)g(s)\,ds = g(t)(\mathrm{H})\int_a^t f(s)\,ds - \int_a^t \left(g'(s)(\mathrm{H})\int_a^s f(\tau)\,d\tau\right) ds, \quad \forall t\in[a,b].$$

Using the Hartman-type function $G$ we can obtain $W^{2,1}_{\mathrm{H},X}([0,1])$-functions:

**Proposition 23.** *Let $f:[0,1]\to X$ be a Henstock integrable function. Then*:

(1) *for every $t\in[0,1]$, $G(t,\cdot)f(\cdot)$ and $\frac{\partial G}{\partial t}(t,\cdot)f(\cdot)$ are Henstock integrable and the function $u_f:[0,1]\to X$ defined by $u_f(t)=(\mathrm{H})\int_0^1 G(t,s)f(s)\,ds$, $\forall t\in[0,1]$ satisfies the following conditions*: $u_f(0)=0$, $u_f(\theta)=u_f(1)$; *and*
(2) *$u_f$ is continuous*;
(3) *$u_f$ is a.e. derivable and its derivative is $\dot u_f(t)=(\mathrm{H})\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds$*;
(4) *$\dot u_f$ is a.e. derivable and its derivative satisfies $\ddot u_f=f$.*

**Proof.** The assertion (1) follows from Lemmas 12 and 22. We can thus define the $X$-valued function $u_f(t)=(\mathrm{H})\int_0^1 G(t,s)f(s)\,ds$, $\forall t\in[0,1]$. By definition, $u_f(\theta)=u_f(1)$ and $u_f(0)=(\mathrm{H})\int_0^1 G(0,s)f(s)\,ds=0$. In order to prove the assertions (2)–(4), consider only the case $t\in[0,\theta[$. Then $u_f(t)=(\mathrm{H})\int_0^t -sf(s)\,ds - t(\mathrm{H})\int_t^\theta f(s)\,ds + t(\mathrm{H})\int_\theta^1 \frac{s-1}{1-\theta}f(s)\,ds$. By Theorem 6, it is a.e. derivable and $\dot u_f(t) = -(\mathrm{H})\int_t^\theta f(s)\,ds + (\mathrm{H})\int_\theta^1 \frac{s-1}{1-\theta}f(s)\,ds = (\mathrm{H})\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds$ and also $\dot u$ is a.e. derivable and $\ddot u_f(t)=f(t)$. □

From Proposition 23 we easily deduce

**Proposition 24.** *Let $f:[0,1]\to X$ be a Henstock integrable function. Then the second order differential equation*

$$\begin{cases} \ddot u(t)=f(t), & \text{a.e. } t\in[0,1],\\ u(0)=0,\ u(\theta)=u(1) \end{cases}$$

*has an unique $W^{2,1}_{\mathrm{H},X}([0,1])$-solution, $u_f(t)=(\mathrm{H})\int_0^1 G(t,s)f(s)\,ds$.*

Finally, we study the existence of $W^{2,1}_{\mathrm{H},X}([0,1])$-solutions for the second order differential inclusion $(*)$. Let us introduce, for the Henstock integral, a notion of integrable boundedness which is similar to those already known for the Bochner, Pettis and Henstock–Kurzweil–Pettis integrals.

**Definition 25.** A measurable multifunction is said to be Henstock integrably bounded if every measurable selection is Henstock-integrable.

**Remark 26.** It is known that a multifunction $F$ is integrably bounded if and only if all measurable selections are Bochner integrable. We have the same result for $\mathcal{P}_{wkc}(X)$-valued multifunctions in the Pettis integrable case [5, Theorem 5.4] and in the Henstock–Kurzweil–Pettis one [12, Theorem 1].

Using characterizations given in Theorem 11 for HKP-integrable multifunctions, we obtain, as in Theorem 21, the following:

**Theorem 27.** *Let $\Gamma:[0,1]\to\mathcal{P}_{kc}(X)$ be measurable, Henstock integrably bounded and $F:[0,1]\times X\times X\to\mathcal{P}_{kc}(X)$ satisfy the following conditions*:

(1) $F(t,x,y)\subset\Gamma(t)$, $\forall t\in[0,1]$, $\forall x,y\in X$;
(2) $F(\cdot,x,y)$ *is measurable, for every* $x,y\in X$;
(3) $F(t,\cdot,\cdot)$ *is upper semi-continuous on* $X\times X$, *for each* $t\in[0,1]$.

*Then the $W^{2,1}_{\mathrm{H},X}([0,1])$-solutions set of differential inclusion* $(*)$ *is nonempty and compact in* $C([0,1],X)$.

**Proof.** Let us note, first of all, that Theorem 11 is valid if we replace everywhere "weakly compact" by "compact." If $\gamma$ is a Henstock integrable selection of $\Gamma$, then we are able to find a compact convex-valued Pettis integrable multifunction $\Gamma_1$ such that $\Gamma(t)=\gamma(t)+\Gamma_1(t)$, for any $t\in[0,1]$ (moreover, since $\Gamma$ is, by hypothesis, Henstock integrably bounded, so is $\Gamma_1$). Using [1, Theorem 3.3], the set-valued function

$$\widetilde{F}:[0,1]\times X\times X\to\mathcal{P}_{kc}(X),$$
$$\widetilde{F}(t,x,y)=-\gamma(t)+F\left(t,x+(\mathrm{H})\int_0^1 G(t,s)\gamma(s)\,ds,\ y+(\mathrm{H})\int_0^1\frac{\partial G}{\partial t}(t,s)\gamma(s)\,ds\right)$$

satisfies that the set of solutions of differential inclusion

$$\begin{cases}\ddot v(t)\in\widetilde{F}\big(t,v(t),\dot v(t)\big), & \text{a.e. } t\in[0,1],\\ v(0)=0,\ v(\theta)=v(1)\end{cases}$$

that are continuous, two times a.e. weakly derivable, the second derivative being Pettis integrable, is nonempty and compact in $C([0,1],X)$ provided with the topology of uniform convergence. But the solutions are obtained by using the Hartman-type function from the Pettis-integrable selections of $\Gamma_1$, that are, in our case, Henstock integrable too. Finally, applying Proposition 23, we deduce, as in Theorem 21, that the $W^{2,1}_{\mathrm{H},X}([0,1])$-solutions set of our differential inclusion $(*)$ is nonempty and $C([0,1],X)$-compact. □

## 5. Continuous dependence on a parameter

We suppose in the sequel that the HKP-integrable function that is governing the three boundary value second order equations in Proposition 14 depends on a parameter $\lambda\in J$, $J$ being a real interval. We are looking for sufficient conditions to ensure the continuity of solution with respect to the parameter.

The first result yields the weak pointwise continuity of solution.

**Proposition 28.** *Let $f:[0,1]\times J\to X$ satisfy the following conditions*:

(1) *for every $\lambda\in J$, $f(\cdot,\lambda)$ is HKP-integrable on* $[0,1]$;
(2) *the function $\lambda\in J\mapsto f(\cdot,\lambda)$ is continuous with respect to the w-HKP topology.*

*Then, denoting by $u_\lambda$ the $W^{2,1}_{\mathrm{HKP},X}([0,1])$-solution of differential equation*

$$\begin{cases} \ddot{u}(t)=f(t,\lambda), & \text{a.e. } t\in[0,1],\\ u(0)=0,\ u(\theta)=u(1), \end{cases}$$

*$(u_{\lambda_n})_n$ weakly pointwise converges to $u_\lambda$ when $\lambda_n\to\lambda$.*

**Proof.** We have already seen that the solution of the previous equation is given by $u_\lambda(t)=(\mathrm{HKP})\int_0^1 G(t,s)f(s,\lambda)\,ds$, for any $t\in[0,1]$.

From hypothesis (2), if $\lambda_n\to\lambda$, then for every $x^*\in X^*$ and every $g$ of bounded variation,

$$(\mathrm{HK})\int_0^1 g(s)\big\langle x^*,f(s,\lambda_n)\big\rangle\,ds\to(\mathrm{HK})\int_0^1 g(s)\big\langle x^*,f(s,\lambda)\big\rangle\,ds.$$

For each $t\in[0,1]$, $G(t,\cdot)$ is of bounded variation, thus, for all $x^*\in X^*$, the sequence $((\mathrm{HK})\int_0^1 G(t,s)\langle x^*,f(s,\lambda_n)\rangle\,ds)_n$ converges to $(\mathrm{HK})\int_0^1 G(t,s)\langle x^*,f(s,\lambda)\rangle\,ds$. So

$$\left\langle x^*,(\mathrm{HKP})\int_0^1 G(t,s)f(s,\lambda_n)\,ds\right\rangle\to\left\langle x^*,(\mathrm{HKP})\int_0^1 G(t,s)f(s,\lambda)\,ds\right\rangle,$$

that is to say that $(u_{\lambda_n})_n$ weakly pointwise converges to $u_\lambda$. $\square$

The proof of the second result will use the following

**Lemma 29.** *Let $f_n:[0,1]\to\mathbb{R}$ be a sequence of HK-integrable functions $\|\cdot\|_A$-convergent to $f$ and let $g$ be a real function of bounded variation. Then the sequence $(gf_n)_n$ (of HK-integrable functions, by Lemma* 3) *converges in the Alexiewicz norm to $gf$.*

**Proof.** By Remark 2, for every $t\in[0,1]$ one has

$$(\mathrm{HK})\int_0^t g(s)f_n(s)\,ds=g(t)(\mathrm{HK})\int_0^t f_n(s)\,ds-\int_0^t\left((\mathrm{HK})\int_0^s f_n(\tau)\,d\tau\right)dg,$$

the last integral being of Riemann–Stieltjes type, and the same is valid for $f$.

As $g$ is of bounded variation, let $M$ be its upper bound. Thus

$$\begin{aligned}\|gf_n-gf\|_A&\leqslant 2\sup_{t\in[0,1]}\left|(\mathrm{HK})\int_0^t g(s)\big(f_n(s)-f(s)\big)\,ds\right|\\ &\leqslant 2M\|f_n-f\|_A+2\sup_{t\in[0,1]}\left|\int_0^t\left((\mathrm{HK})\int_0^s f_n(\tau)-f(\tau)\,d\tau\right)dg\right|.\end{aligned}$$

In order to simplify the calculus, let us denote by $\tilde{f}$ (respectively $\tilde{f_n}$) the primitive of $f$ (respectively $f_n$). The convergence of $f_n$ to $f$ in the Alexiewicz norm can be expressed in the terms of their primitives, by $\tilde{f_n}$ converges uniformly to $\tilde{f}$. So, for every $\varepsilon > 0$, we can find $n_\varepsilon \in \mathbb{N}$ such that, for all $n \geqslant n_\varepsilon$ and $t \in [0,1]$, $|\tilde{f_n}(t) - \tilde{f}(t)| \leqslant \varepsilon$. Therefore, by the definition of Riemann–Stieltjes integral, for each $n \geqslant n_\varepsilon$ and $t \in [0,1]$, we have $|\int_0^t (\tilde{f_n} - \tilde{f})\,dg| \leqslant \varepsilon V(g)$, where $V(g)$ denotes the total variation of $g$.

Consequently, $\|gf_n - gf\|_A \leqslant 2M\|f_n - f\|_A + 2\varepsilon V(g) \leqslant \varepsilon(2M + 2V(g))$ for every $n \geqslant n_\varepsilon$, and then the sequence $(gf_n)_n$ is $\|\cdot\|_A$-convergent to $gf$. $\square$

**Proposition 30.** *Let $f : [0,1] \times J \to X$ satisfy hypothesis* (1) *in Proposition* 28 *and* (2′) *for every $x^* \in X^*$, the function $\lambda \mapsto \langle x^*, f(\cdot,\lambda)\rangle$ is continuous on $J$ with respect to the Alexiewicz norm topology. Then the sequence $(u_{\lambda_n})_n$ converges in the topology of $C([0,1], X_w)$ to $u_\lambda$ when $\lambda_n \to \lambda$.*

**Proof.** We show only that $(u_{\lambda_n})_n$ converges uniformly (with respect to the weak topology on $X$) to $u_\lambda$ on $[0,\theta[$. By definition,

$$\begin{aligned}\big\langle x^*, u_{\lambda_n}(t) - u_\lambda(t)\big\rangle &= (\mathrm{HK})\int_0^1 G(t,s)\big\langle x^*, f(s,\lambda_n) - f(s,\lambda)\big\rangle\,ds\\ &= (\mathrm{HK})\int_0^t -s\big\langle x^*, f(s,\lambda_n) - f(s,\lambda)\big\rangle\,ds\\ &\quad - t(\mathrm{HK})\int_t^\theta \big\langle x^*, f(s,\lambda_n) - f(s,\lambda)\big\rangle\,ds\\ &\quad + t(\mathrm{HK})\int_\theta^1 \frac{s-1}{1-\theta}\big\langle x^*, f(s,\lambda_n) - f(s,\lambda)\big\rangle\,ds,\end{aligned}$$

so the uniform convergence of $(u_{\lambda_n})_n$ to $u_\lambda$ follows from the previous lemma. $\square$

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